Study on stationary solutions to the problem of phytoplankton dynamics considering transformation of phosphorus, nitrogen and silicon compounds

Yu.V. Belova¹, A.M. Atayan², A.E. Chistyakov³, A.V. Strazhko**
¹,²,³,⁴ Don State Technical University, Rostov-on-Don, Russian Federation

Introduction. The solution to the problem of transformation of phosphorus, nitrogen and silicon forms is studied. This problem arises under modeling phytoplankton dynamics in shallow-water bodies including the Azov Sea. The phytoplankton dynamics model is formulated as a boundary value problem for the system of diffusion-convection-response equations and takes into account the absorption and release of nutrients by phytoplankton, as well as the transition of nutrients from one compound to another. To calculate the initial conditions and parameters of the equations under which the steady-state regime occurs, the software is developed, which is based on the model describing changes in phytoplankton concentrations without considering current effects. This model is represented by a system of inhomogeneous differential equations. Based on the developed software, the initial conditions and parameters of the phytoplankton dynamics model in the Azov Sea are calculated experimentally.

Materials and Methods. A 3D model of phytoplankton dynamics is considered taking into account the transformation of phosphorus, nitrogen and silicon compounds based on the system of nutrient transport equations. The case of a spatially uniform distribution of substances is considered to specify the parameters of the model at which the stationary modes occur. Because of simplification, a system of ordinary differential equations solved through the Runge-Kutta method is obtained. Research Results. The software is developed to specify the initial conditions and parameters of the phytoplankton dynamics model considering the transformation of phosphorus, nitrogen and silicon compounds. Several numerical experiments are performed under the assumption that the development of phytoplankton is limited by...
a single biogenic substance. As a result of the computational experiment, it can be seen that with the obtained values of the initial concentrations and parameters of the equations, stationary modes occur for the system of ordinary differential equations describing the case of the spatially uniform distribution of substances.

Discussion and Conclusions. The mathematical model of the transformation of phosphorus, nitrogen and silicon forms in the problem of phytoplankton dynamics is studied. Stationary modes for the system of ordinary differential equations are obtained, for which the values of the system parameters and initial conditions are determined. The results obtained can be used in further simulation of the phytoplankton dynamics considering the transformation of phosphorus, nitrogen and silicon compounds with account for convection-diffusion, salinity, and temperature.


Introduction. Because of the development of major cities on the coast of shallow water bodies and river systems that flow into these water bodies, eutrophication has become more frequent. The growth of algae in reservoirs is caused by an increase in the flow of nitrogen and phosphorus compounds from the adjacent land areas. Each water body is unique and requires a thorough study. Field investigations [1] and mathematical modeling are used to explore water bodies. Without downplaying the role of field experiments, we can say that mathematical modeling is less costly, and it allows us to predict the behavior of the ecosystem.

To study the Sea of Azov, a three-dimensional model of hydrodynamics [2, 3] including the equations of motion in three spatial directions was developed. In [4], this model was made for the case of dynamic rebuilding of the computational domain geometry due to the tidal effects. The investigation of this model accuracy is given in [5]. In [6–8], the reconstruction of an ecological catastrophe that occurred in 2001 caused by an excessive concentration of algae in the eastern part of the Sea of Azov is given. In [9], methods of controlling the suffocation phenomena arising in the Sea of Azov were proposed. The [10–12] papers are devoted to studying the dynamics of phyto- and zooplankton.

The water condition in shallow water bodies is changing rapidly, and mathematical models need to be refined. The parameters determination of the three-dimensional model of the phyto- and zooplankton dynamics is laborious; therefore, it is proposed to use a simplified model to calculate these parameters.

The work objective is to improve the parameters of the model of the phytoplankton dynamics considering the transformation of phosphorus, nitrogen and silicon compounds, under which stationary regimes occur with the assumption of a spatially uniform distribution of substances.

Materials and Methods. The model is based on the system of equations for the transport of nutrients [15, 16], the form of which for each $F_i$ model block is

**Keywords**: phytoplankton, phosphorus, nitrogen, silicon, biogenic, chemical-biological source, convection-diffusion-response equation, Cauchy problem for system of ordinary differential equations, stationary mode.

**Key words**: фитопланктон, фосфор, азот, кремний, биоген, химико-биологический источник, уравнение конvection-диффузии-реакции, задача Коши для системы обыкновенных дифференциальных уравнений, стационарный режим.

\[
\frac{\partial q_i}{\partial t} + u \frac{\partial q_i}{\partial x} + v \frac{\partial q_i}{\partial y} + w \frac{\partial q_i}{\partial z} = \text{div}(k \text{ grad } q_i) + R_s,
\]

where \( q_i \) is concentration of the \( i \)-th component, [mg/l]; \( i \in M = \{ F_1, F_2, F_3, PO_4, POP, DOP, NO_3, NO_2, NH_4, Si \} \); \( \{u, v, w\} \) are components of the velocity vector of the water flow, [m/s]; \( k \) is turbulent exchange coefficient, [m^2/s]; \( R_s \) is function-source of nutrients, [mg/l \cdot s].

In equation (1), \( i \) index indicates the type of substance (Table 1).

### Table 1

<table>
<thead>
<tr>
<th>No.</th>
<th>Notation</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( F_1 )</td>
<td>Chlorella vulgaris green algae</td>
</tr>
<tr>
<td>2</td>
<td>( F_2 )</td>
<td>Aphanizomenon flos-aquae green-blue algae</td>
</tr>
<tr>
<td>3</td>
<td>( F_3 )</td>
<td>Sceletonema costatum diatom</td>
</tr>
<tr>
<td>4</td>
<td>PO_4</td>
<td>phosphates</td>
</tr>
<tr>
<td>5</td>
<td>POP</td>
<td>suspended organic phosphor</td>
</tr>
<tr>
<td>6</td>
<td>DOP</td>
<td>soluble organic phosphor</td>
</tr>
<tr>
<td>7</td>
<td>NO_3</td>
<td>nitrates</td>
</tr>
<tr>
<td>8</td>
<td>NO_2</td>
<td>nitrates</td>
</tr>
<tr>
<td>9</td>
<td>NH_4</td>
<td>ammonium</td>
</tr>
<tr>
<td>10</td>
<td>Si</td>
<td>soluble inorganic silica (silicic acids)</td>
</tr>
</tbody>
</table>

Chemical and biological reactions are described by the following equations

\[
R_{pF} = \sum_{i=1}^{10} s_i F_i \left( K_{pF} - 1 \right) q_F + K_{pF} q_F,
\]

\[
R_{sF} = \sum_{i=1}^{10} s_i F_i \left( K_{sF} - 1 \right) q_F + K_{sF} q_F + K_{pF} q_F.
\]

\[
R_{nF} = \sum_{i=1}^{10} s_i F_i \left( K_{nF} - 1 \right) q_F + K_{nF} q_F + K_{pF} q_F.
\]

where \( K_{pF} \) is specific breathing rate of phytoplankton; \( K_{sF} \) is specific die-off rate of phytoplankton; \( K_{nF} \) is specific excretion rate of phytoplankton; \( K_{pF} \) is specific rate of POP autolysis; \( K_{pF} \) is coefficient of POP phosphatification; \( K_{pF} \) is coefficient of DOP phosphatification; \( K_{pF} \) is specific rate of ammonium oxidation to nitrites under nitrification; \( K_{pF} \) is specific rate of nitrite oxidation to nitrates under nitrification; \( s_F, s_F, s_F \) are normalization coefficients between the content of \( N, P, Si \) in organic matter.

Phytoplankton growth rate is determined by the following expressions:

\[
C_{pF} = K_{pF} \min \left\{ f_F \left( q_{PO_4} \right), f_F \left( q_{PO_4}, q_{PO_4}, q_{PO_4} \right) \right\},
\]

\[
C_{nF} = K_{nF} \min \left\{ f_F \left( q_{PO_4} \right), f_F \left( q_{PO_4}, q_{PO_4}, q_{PO_4} \right), f_F \left( q_{PO_4} \right) \right\},
\]

where \( K_{pF} \) is maximum specific phytoplankton growth rate.
Fig. 1. Model scheme of biogeochemical transformation of phosphorus, nitrogen and silica forms

Functions describing nutrient content

- for phosphorus: \( f_q(q_{\text{PO}_4}) = \frac{q_{\text{PO}_4}}{q_{\text{PO}_4} + K_{\text{PO}_4}} \),

where \( K_{\text{PO}_4} \) is phosphates half saturation constant;

- for silica: \( f_q(q_{\text{Si}}) = \frac{q_{\text{Si}}}{q_{\text{Si}} + K_{\text{Si}}} \),

where \( K_{\text{Si}} \) is silica half saturation constant;

- for nitrogen: \( f_q(q_{\text{NO}_3}, q_{\text{NH}_4}, q_{\text{NH}_3}) = f_n^{(1)}(q_{\text{NO}_3}, q_{\text{NH}_4}, q_{\text{NH}_3}) + \frac{f_n^{(2)}(q_{\text{NH}_3})}{\frac{q_{\text{NH}_4}}{K_{\text{NH}_4} + q_{\text{NH}_4} + K_{\text{NH}_3}} + \frac{q_{\text{NH}_3}}{K_{\text{NH}_3} + q_{\text{NH}_3}}}, \)

where \( K_{\text{NH}_3} \) is nitrates half saturation constant, \( K_{\text{NH}_4} \) is ammonium half saturation constant, \( K_{\text{NH}_4} \) is ammonium inhibition ratio.

For the system (1), it is necessary to specify the vector field of the water flow velocities, as well as \( q_i \) initial values of the concentration functions

\[
q_i(x, y, z, 0) = q^0_i(x, y, z), \quad (x, y, z) \in G, \quad t = 0, \quad i \in M. \tag{2}
\]

Assume \( \Sigma \) boundary of \( G \) cylindrical region is piecewise smooth and \( \Sigma = \Sigma_R \cup \Sigma_S \cup \sigma \), where \( \Sigma_R \) is surface of the reservoir bottom, \( \Sigma_S \) is still water surface, \( \sigma \) is lateral (cylindrical) surface. Suppose \( u_i \) is vector component of the water flow velocity normal to \( \Sigma \), and \( n \) is outward normal vector to \( \Sigma \). For \( q_i \) concentrations, we assume:

- on side boundary:

\[
q_i = 0, \quad \text{on } \sigma, \quad \text{if } u_i < 0, \quad i \in M; \tag{3}
\]

\[
\frac{\partial q_i}{\partial n} = 0, \quad \frac{\partial q_i}{\partial n} = 0, \quad \text{on } \sigma, \quad \text{if } u_i \geq 0, \quad i \in M;
\]

- on \( \Sigma_R \) there is reservoir surface:

\[
\frac{\partial q_i}{\partial z} = 0, \quad i \in M; \tag{5}
\]

- on the bottom \( \Sigma_R \):

\[
k \frac{\partial q_i}{\partial z} = \varepsilon_i q_i, \quad i \in \{ F_1, F_2, F_3 \},
\]
where \( \varepsilon_{ij} \) and \( \varepsilon_{ij} \) are sedimentation rates of algae and nutrients to the bottom.

**Stationary Mode.** Consider the case of a spatially uniform distribution of substances (phytoplankton, forms of phosphorus, nitrogen and silica); then each of the equations (1) is simplified; and as a result, we get the following system of ordinary differential equations (ODE):

\[
\frac{dC_i}{dt} = C_i \left(1 - K_{r, i} \right) q_i - K_{r, i} q_i - K_{r, i} q_i , \quad i = 1, 3 ,
\]

\[
\frac{dq_{\text{pop}}}{dt} = \sum_{ij} s_{ij} K_{r, ij} q_i - K_{r, ij} q_{\text{pop}} - K_{r, ij} q_{\text{pox}},
\]

\[
\frac{dq_{\text{pox}}}{dt} = \sum_{ij} s_{ij} K_{r, ij} q_i + K_{r, ij} q_{\text{pox}} - K_{r, ij} q_{\text{slo}},
\]

\[
\frac{dq_{\text{slo}}}{dt} = \sum_{ij} s_{ij} C_i \left(K_{r, ij} - 1 \right) q_i + K_{r, ij} q_{\text{pox}} + K_{r, ij} q_{\text{slo}},
\]

\[
\frac{dq_{\text{sl}}}{dt} = \sum_{ij} s_{ij} C_i \left(K_{r, ij} - 1 \right) q_i - K_{r, ij} q_i - K_{r, ij} q_i ,
\]

\[
\frac{dq_{\text{slo}}}{dt} = \sum_{ij} s_{ij} C_i \left(K_{r, ij} - 1 \right) q_i + K_{r, ij} q_{\text{pox}} + K_{r, ij} q_{\text{slo}} ,
\]

\[
\frac{dq_{\text{sl}}}{dt} = \sum_{ij} s_{ij} C_i \left(K_{r, ij} - 1 \right) q_i + s_{ij} K_{r, ij} q_i .
\]

We solve the system of ordinary differential equations by the Runge–Kutta method [15–17]. We will conduct several numerical experiments, assuming that the development of phytoplankton depends on a single limiting substance.

**Research Results.** For the ODE system (7), we calculate the initial conditions and parameters of the equations at which the stationary regimes occur. Let us take the initial concentration values: \( q_i \left(0\right) = 2.5 \text{ mg/l} \), \( q_{\text{pox}} \left(0\right) = 2.6 \text{ mg/l} \), \( q_{\text{sl}} \left(0\right) = 0.91 \text{ mg/l} \), \( q_{\text{sl}} \left(0\right) = 0.07 \text{ mg/l} \), \( q_{\text{pox}} \left(0\right) = 0.07 \text{ mg/l} \), \( q_{\text{sl}} \left(0\right) = 0.005 \text{ mg/l} \), \( q_{\text{sl}} \left(0\right) = 0.11 \text{ mg/l} \), \( q_{\text{slo}} \left(0\right) = 0.0178 \text{ mg/l} \), \( q_{\text{slo}} \left(0\right) = 0.304 \text{ mg/l} \), \( q_{\text{slo}} \left(0\right) = 0.4 \text{ mg/l} \); coefficients: \( K_{\text{sr}} = 2.8 \text{ day}^{-1} \), \( K_{r, ij} = 0.15 \text{ day}^{-1} \), \( K_{r, ij} = 0.05 \text{ day}^{-1} \), \( K_{r, ij} = 0.15 \text{ day}^{-1} \), \( K_{r, ij} = 0.015 \text{ day}^{-1} \), \( K_{r, ij} = 0.02 \text{ day}^{-1} \), \( K_{r, ij} = 0.1 \text{ day}^{-1} \), \( K_{r, ij} = 0.9 \text{ day}^{-1} \), \( K_{r, ij} = 2.5 \text{ day}^{-1} \), \( K_{r, ij} = 1.46 \text{ day}^{-1} \), \( s_{\text{p}} = 0.01 \), \( s_{\text{N}} = 0.016 \), \( s_{\text{Si}} = 0.023 \), \( K_{r, ij} = 0.024 \), \( K_{r, ij} = 3.0 \), \( K_{r, ij} = 2.0 \), \( K_{r, ij} = 3.0 \).

The obtained stationary modes of the ODE system (7) on the assumption that the development of phytoplankton is limited by a single nutrient (phosphorus, nitrogen or silica) are shown in Fig. 2–4, respectively. Fig. 2 describes the effect of phosphorus on the development of various phytoplankton species; Fig. 3 describes the effect of nitrogen on the development of various phytoplankton species; Fig. 4 describes the effect of nitrogen on diatom development.
Fig. 2. Stationary mode of ODE system under assumption that phytoplankton development is limited by phosphorus: a) green algae (ChV), b) green-blue algae (AF-A), c) diatom (SC), d) suspended organic phosphorus (POP), e) soluble organic phosphorus (DOP), f) phosphates (PO$_4$).
Fig. 3. Stationary mode of ODE system under assumption that phytoplankton development is limited by nitrogen: a) green algae (ChV), b) green-blue algae (AF-A), c) diatom (SC), d) ammonium (NH₄), e) nitrites (NO₂), f) nitrates (NO₃).

Fig. 4. Stationary mode of ODE system under assumption that phytoplankton (diatoms) development is limited by silica: a) diatom (SC), b) silica (Si).
The result of the computational experiment shows that with the above values of the initial concentrations and parameters of the equations, stationary modes occur for the ODE system (7), which describes the case of a spatially uniform distribution of substances. The obtained values will be used in further simulation of the spatially inhomogeneous distribution of substances, saltiness and temperature considering the movement of the aquatic environment [18].

**Conclusion.** A mathematical model of the transformation of forms of phosphorus, nitrogen and silica in the problem of phytoplankton dynamics is studied in the paper. The case of spatially uniform distribution of substances (phytoplankton, forms of phosphorus, nitrogen and silica) is considered. The system is divided into three systems of ordinary differential equations, each of which simulates the dependence of phytoplankton growth on a single nutrient. These systems are solved by the Runge-Kutta method (Fig. 2–4); stationary modes are obtained, for which the values of the system parameters and initial conditions are determined.

The results obtained will be used for the further simulation of the phytoplankton dynamics considering the transformation of phosphorus, nitrogen and silica compounds, taking into account diffusion-convection, saltiness, and temperature.

**References**


Received 20.11.2018
Submitted 21.12.2018
Scheduled in the issue 11.01.2019

Authors:

Belova, Yuliya V.,
Junior research scholar, Research Institute for Mathematical Modeling and Forecasting of Complex Systems, Don State Technical University (1, Gagarin sq., Rostov-on-Don, 344000, RF),
ORCID: https://orcid.org/0000-0002-2639-7451
yvbelova@yandex.ru

Atayan, Asya M.,
teaching assistant of the Computer and Automated Systems Software Department, Don State Technical University (1, Gagarin sq., Rostov-on-Don, 344000, RF),
ORCID: https://orcid.org/0000-0003-4629-1002
atayan24@yandex.ru

Chistyakov, Aleksandr E.,
professor of the Computer and Automated Systems Software Department, Don State Technical University (1, Gagarin sq., Rostov-on-Don, 344000, RF), Dr.Sci. (Phys.-Math.), professor,
ORCID: https://orcid.org/0000-0002-8323-6005
cheese_05@mail.ru

Strazhko, Aleksandr V.,
student of the Computer and Automated Systems Software Department, Don State Technical University (1, Gagarin sq., Rostov-on-Don, 344000, RF),
ORCID: https://orcid.org/0000-0002-2449-8531
strajcko2@gmail.com